The Fourier transform of the quadratic Kubo-Bass term

The behavior given by the first two terms of the Kubo-Bass series is

$$B(t) = \int R(t')G_1(t-t')dt' + \int\int R(t')R(t'')G_2(t-t',t'-t'')dt'dt'' \quad . \tag{1}$$

While the Fourier transform of the linear term is well known, $r(f)g_1(f)$, the quadratic term has rarely been used. In the frequency domain the $2^{\underline{\text{nd}}}$ order term is given by

$$b_2(f) = \int e^{2\pi i f t}dt \int\int R(t')R(t'')G_2(t-t',t'-t'')dt'dt'' \quad . \tag{2}$$

Replacement of $R(t')$ and $R(t'')$ by their inverse Fourier transform representation[1] yields

$$\begin{aligned} b_2(f) & \\ = & \int e^{2\pi i f t}dt \int\int \left[\int r(f')e^{-2\pi i f't'}df'\right]\left[\int r(f'')e^{-2\pi i f''t''}df''\right] G_2(t-t',t'-t'')dt'dt'' \quad , && (3)\\ = & \int dt \int dt' \int dt'' \int df' \int df'' e^{2\pi i(ft-f't'-f''t'')}G_2(t-t',t'-t'')r(f')r(f'') \quad . && (4) \end{aligned}$$

Set $v = t'-t''$ or $t'' = t'-v$ and $dt'' = -dv$, so that upon substitution and rearrangement of the integration order

$$\begin{aligned} b_2(f) & \\ = & -\int dt \int dt' \int df' \int df'' e^{2\pi i f t}\left[\int dv\, e^{-2\pi i(f't'+f''(t'-v))}G_2(t-t',v)\right] r(f')r(f'') \quad , && (5)\\ = & -\int dt \int dt' \int df' \int df'' e^{2\pi i f t}\left[\int dv\, e^{-2\pi i(f't'+f''t'-f''v)}G_2(t-t',v)\right] r(f')r(f'') \quad , && (6)\\ = & -\int dt \int dt' \int df' \int df'' e^{2\pi i f t}e^{-2\pi i(f'+f'')t'}\left[\int dv\, e^{2\pi i f''v}G_2(t-t',v)\right] r(f')r(f'') \quad . && (7) \end{aligned}$$

Next set $u = t-t'$ or $t' = t-u$ and $dt' = -du$, hence upon substitution and further rearrangement

$$\begin{aligned} b_2(f) & \\ = & \int dt \int df' \int df'' \int du\, e^{2\pi i f t}e^{-2\pi i(f'+f'')(t-u)}\left[\int dv\, e^{2\pi i f''v}G_2(u,v)\right] r(f')r(f'') \quad , && (8)\\ = & \int dt \int df' \int df'' \int du\, e^{2\pi i(ft-f't+f'u-f''t+f''u)}\left[\int dv\, e^{2\pi i f''v}G_2(u,v)\right] r(f')r(f'') \quad , && (9)\\ = & \int dt \int df' \int df'' \int du\, e^{2\pi i(f-f'-f'')t}e^{2\pi i(f'+f'')u}\left[\int dv\, e^{2\pi i f''v}G_2(u,v)\right] r(f')r(f'') \quad , && (10)\\ = & \int df' \int df'' \int dt\, e^{2\pi i(f-f'-f'')t}\left[\int\int du\, dv\, e^{2\pi i(f'+f'')u}e^{2\pi i f''v}G_2(u,v)\right] r(f')r(f'') \quad . && (11) \end{aligned}$$

The factor in the square parentheses is a two-dimensional Fourier transform, so Equation 11 can be rewritten in the simpler form of

$$b_2(f) = \int df' \int df'' \underbrace{\int dt\, e^{2\pi i(f-f'-f'')t}}\, g_2(f'+f'', f'')r(f')r(f'') \quad . \tag{12}$$

Recognizing the bracketed factor in Equation 12 as the Fourier transform representation of $\delta((f-f')-f'')$[2] and, that from symmetry it is also equal to $\delta(f''-(f-f'))$, yields upon substitution and evaluation of the integral over df''

$$b_2(f) = \int df' \int df'' \delta(f''-(f-f'))g_2(f'+f'', f'')r(f')r(f'') \quad , \tag{13}$$

$$= \int df' g_2(f'+(f-f'), f-f')r(f')r(f-f') \quad , \tag{14}$$

$$= \int df' g_2(f, f-f')r(f')r(f-f') \quad . \tag{15}$$

All that remains are the two small tasks of developing a stable numerical technique to estimate $g_2(f, f-f')$ from data and an experimental test of whether Equation 15 can outperform linear term.

References

[1] Ron Bass both showed the value of replacing $R(t')$ and $R(t'')$ by their inverse Fourier transform representation and developed the general outline of this calculation in the early 1980's. This calculation is just a recovery of his original result.

[2] Arfken, G. (1970). *Mathematical Methods for Physicists*, (pp. 671–673). New York:Academic Press.